Topological black holes dressed with a conformally coupled scalar field and electric charge

Cristián Martínez[*] and Ricardo Troncoso[†]

Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile.

Juan Pablo Staforelli[‡]

Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile.

Departamento de Física, P. Universidad Católica de Chile, Casilla 306, Santiago 22, Chile.

Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile.

Abstract

Electrically charged solutions for gravity with a conformally coupled scalar field are found in four dimensions in the presence of a cosmological constant. If a quartic self-interaction term for the scalar field is considered, there is a solution describing an asymptotically locally AdS charged black hole dressed with a scalar field that is regular on and outside the event horizon, which is a surface of negative constant curvature. This black hole can have negative mass, which is bounded from below for the extremal case, and its causal structure shows that the solution describes a "black hole inside a black hole". The thermodynamics of the non-extremal black hole is analyzed in the grand canonical ensemble. The entropy does not follow the area, and there is an effective Newton constant which depends on the value of the scalar field at the horizon. If the base manifold is locally flat, the solution has no electric charge, and the scalar field has a vanishing stress-energy tensor so that it dresses a locally AdS spacetime with a nut at the origin. In the case of vanishing selfinteraction, the solutions also dress locally AdS spacetimes, and if the base manifold is of negative constant curvature a massless electrically charged hairy black hole is obtained. The thermodynamics of this black hole is also analyzed. It is found that the bounds for the black holes parameters in the conformal frame obtained from requiring the entropy to be positive are mapped into the ones that guarantee cosmic censorship in the Einstein frame.

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[*]Electronic address: martinez@cecs.cl

[†]Electronic address: ratron@cecs.cl

[‡]Electronic address: jp@cecs.cl

I. INTRODUCTION

In four-dimensional asymptotically flat spacetimes, only horizons with the topology of a sphere are compatible with a well defined causal structure for black holes [1]. However, this topological censorship can be circumvented introducing a negative cosmological constant and a number of black holes with flat or hyperbolic horizons have been reported in four and higher dimensions [2, 3, 4, 5, 6, 7]. Moreover, this class of black holes are found in gravity theories containing higher powers of the curvature [8, 9, 10, 11]. The black hole thermodynamics is drastically affected by the topology of the event horizon. In fact, the Hawking-Page phase transition for the Schwarzschild-AdS black hole [12] does not occur for asymptotically locally anti-de Sitter black holes whose horizons have vanishing or negative constant curvature, and they are thermodynamically –locally– stable since they have a positive heat capacity [6]. The inclusion of electric charge further modifies the thermodynamical properties of black holes and a more complex phase structure arises, allowing for an analogous of the Hawking-Page phase transition [5, 7, 13, 14]. Further developments concerning black holes with nontrivial topology can be found in Refs. [15].

On the other hand, one could naively think that the simplest source of matter for a black hole would corresponds to a real scalar field. However, the so called no-hair conjecture [16, 17, 18], which originally stated that a black hole should be characterized only in terms of its mass, angular momentum and electric charge, indicates that one should rule out this kind of hairy black hole solutions (for recent discussions see e.g., [19]). Nonetheless, this conjecture can be circumvented in different ways. For instance, exact black hole solutions with minimally and conformally coupled self-interacting scalar fields have been found in three [20, 21] and four dimensions [22, 23, 24] in the presence of a cosmological constant Λ. In the case of vanishing Λ, a four-dimensional black hole is also known [25], but the scalar field diverges at the horizon. Numerical hairy black hole solutions of this sort have also been found in Refs. [26, 27, 28, 29, 30, 31].

In this article, four-dimensional electrically charged solutions for the Einstein-Maxwell action with a conformally coupled scalar field and a cosmological constant are found. A quartic self-interaction term for the scalar field, which does not break the conformal invariance of its field equation is considered. Solutions with a nonvanishing selfinteraction coupling constant are discussed in section III, where the base manifold is assumed to be a

surface of constant curvature. In the spherically symmetric case, the solution reduces to the one found in Ref. [22], which describes a hairy black hole provided the cosmological constant is non negative. If the base manifold is a surface of negative constant curvature, the solution corresponds to an asymptotically locally AdS charged black hole dressed with a scalar field that is regular on and outside the event horizon. This black hole can have negative mass, which is bounded from below for the extremal case, and its causal structure shows that the solution describes a "black hole inside a black hole". The case of a vanishing electric charge, reduces to the solution found in Ref. [23], for which some novel interesting features have been found.[1] The thermodynamics of the non-extremal black hole is analyzed in the grand canonical ensemble, and it is shown that the entropy does not follow the area law. If the base manifold is locally flat, the solution is electrically neutral, and the scalar field has a vanishing stress-energy tensor so that it dresses a locally AdS spacetime with a nut at the origin.

Section IV is devoted to analyze the case of vanishing self-interaction coupling constant. Nontrivial solutions are found, which curiously have a vanishing energy momentum tensor. The most interesting case is the one for negative cosmological constant with a base manifold of negative constant curvature. In this case the solution corresponds to a locally AdS spacetime, and describes a massless electrically charged hairy black hole is obtained, and its thermodynamics is also analyzed. Finally, in section V, the mapping of these solutions to the Einstein frame, where the scalar field becomes minimally coupled, is discussed. It is found that the bounds for the black holes parameters in the conformal frame which are obtained from requiring the entropy to be positive are mapped into the ones that guarantee cosmic censorship in the Einstein frame.

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[1] For this hairy black hole, there is a critical temperature below which a black hole in vacuum undergoes a spontaneous dressing up with a nontrivial scalar field. Moreover, in this case they only admit a finite number of quasinormal modes [32]. The solution is also expected to be stable against perturbations [30], [33], and it can be uplifted to a solution of eleven-dimensional supergravity [34].

II. EINSTEIN-MAXWELL THEORY WITH A CONFORMALLY COUPLED SELF-INTERACTING SCALAR FIELD

We consider the Einstein-Maxwell system in four dimensions with a cosmological constant Λ and a real conformally coupled self-interacting scalar field, described by the action

$$
I[g_{\mu\nu}, \phi, A_\mu] = \int d^4x \sqrt{-g} \left[\frac{R - 2\Lambda}{16\pi G} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} R\phi^2 - \alpha\phi^4 \right]
$$
$$
- \frac{1}{16\pi} \int d^4x \sqrt{-g} F^{\mu\nu} F_{\mu\nu}, \tag{1}
$$

where G is the Newton's constant and α is an arbitrary coupling constant. The corresponding field equations are[2]

$$
G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G (T^\phi_{\mu\nu} + T^{\text{em}}_{\mu\nu}), \tag{2}
$$
$$
\Box \phi = \frac{1}{6} R\phi + 4\alpha\phi^3, \tag{3}
$$
$$
\partial_\nu (\sqrt{-g} F^{\mu\nu}) = 0, \tag{4}
$$

where $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$, and the energy-momentum tensor is given by the sum of the scalar field contribution

$$
T^\phi_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{6} [g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}]\phi^2 - g_{\mu\nu}\alpha\phi^4, \tag{5}
$$

and the one for the electromagnetic field

$$
T^{\text{em}}_{\mu\nu} = \frac{1}{4\pi} \left(F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\gamma\alpha} F_{\delta\beta} g^{\gamma\delta} \right) g^{\alpha\beta}. \tag{6}
$$

Note that Eq. (3) can be obtained from Eqs. (2) and (4) using the conservation of the energy-momentum tensor, and since the scalar field is conformally coupled, which means that the total energy-momentum tensor is traceless, the Ricci scalar curvature is constant, $R = 4\Lambda$.

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[2] It can be shown that the inclusion of the electromagnetic field does not spoil the duality symmetry described in Ref. [35].

III. SOLUTIONS WITH A NONVANISHING SELF-INTERACTION COUPLING CONSTANT ($\alpha \neq 0$)

The field equations (2), (3) and (4) admit an exact static solution whose metric is given by

$$ds^2 = -\left[-\frac{\Lambda r^2}{3} + \gamma\left(1 + \frac{G\mu}{r}\right)^2\right]dt^2 + \left[-\frac{\Lambda r^2}{3} + \gamma\left(1 + \frac{G\mu}{r}\right)^2\right]^{-1}dr^2 + r^2 d\sigma^2, \quad (7)$$

with $-\infty < t < \infty$ and $r > 0$. Here $d\sigma^2$ stands for the line element of the two-dimensional base manifold Σ which is assumed to be compact, without boundary, and of constant curvature γ that can be normalized to $\pm 1, 0$. This means that the surface Σ is locally isometric to the sphere S^2, flat space $\mathbb{R}^2$, or to the hyperbolic manifold H^2 for $\gamma = +1, 0, -1$, respectively.

The scalar field reads

$$\phi = \sqrt{\frac{-\Lambda}{6\alpha}} \frac{G\mu}{r + G\mu}, \quad (8)$$

which is real provided α and Λ have opposite signs, and the electromagnetic potential is given by

$$A = -\frac{q}{r}dt . \quad (9)$$

The integration constants q and μ are not independent since they must satisfy

$$q^2 = \gamma G\mu^2\left(1 + \frac{2\pi\Lambda G}{9\alpha}\right) . \quad (10)$$

As it it shown below, they are related to the electric charge Q and the mass M, respectively as

$$M = -\gamma\frac{\sigma}{4\pi}\mu \quad \text{and} \quad Q = \frac{\sigma}{4\pi}q, \quad (11)$$

where σ is the area of the base manifold Σ.

For $\gamma = 1$, the only sensible choice for the base manifold Σ is the sphere S^2, since the other smooth manifold available is the real projective space $\mathbb{R}P^2$, which however, is non orientable. This case has been discussed in Ref. [22], where it was shown that the solution describes a hairy black hole provided the cosmological constant is non negative. In the vanishing cosmological constant limit, the self coupling constant must vanish also, and the solution of Ref. [25] is recovered. The thermodynamics and the stability for the black hole with positive cosmological constant have been discussed in Refs. [36] and [37], respectively. Further aspects of this class of solutions have been studied in [38]. The remaining cases, $\gamma = -1, 0$, are analyzed below.

A. Electrically charged hairy black hole with hyperbolic horizon $(\gamma = -1)$

Here we consider the case when the base manifold is of negative constant curvature $\gamma = -1$. This means that Σ must be of the form $\Sigma = H^2/\Gamma$, where Γ is a freely acting discrete subgroup of $O(2,1)$. This manifold has genus $g \geq 2$, and its area is given by

$$\sigma = 4\pi(g-1) \ .$$

In this case, when the cosmological constant is negative $(\Lambda = -3l^{-2})$, the metric (7) takes the form

$$ds^2 = -\left[\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r}\right)^2\right]dt^2 + \left[\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r}\right)^2\right]^{-1}dr^2 + r^2 d\sigma^2, \tag{12}$$

and describes an asymptotically locally AdS black hole. There is a unique curvature singularity at the origin $(r = 0)$. The electromagnetic field is given by (9), and the scalar field

$$\phi = \sqrt{\frac{1}{2\alpha l^2}}\frac{G\mu}{r + G\mu}, \tag{13}$$

has a simple pole at $r = -G\mu$ only if μ is negative, and it is real for $\alpha > 0$.

The components of the total energy-momentum tensor, corresponding to addition of (5) and (6) are

$$T^\mu_\nu = \frac{G\mu^2}{8\pi r^4}\mathrm{diag}(-1,-1,1,1) \ , \tag{14}$$

which satisfies the dominant and strong energy conditions. In fact, the energy-momentum tensor is of type I in the classification of Ref. [39].

Equation (10) fixes the charge-to-mass ratio of the black hole, as a function of the fundamental constants appearing in the action: G, Λ and α, and determines an upper bound for the self-interaction coupling constant

$$0 < \alpha \leq \frac{2\pi G}{3l^2} \ . \tag{15}$$

This bound is saturated when the electric charge vanishes, and in this case one recovers the neutral solution discussed in the appendix of Ref. [23]. It is worth pointing out that, since the Ricci scalar is constant, the scalar field equation (3) has a "mexican hat" shaped effective self interaction potential, whose mass term is given by $-2l^{-2}$, satisfying the Breitenlohner-Freedman bound [40], which has been shown to hold also for the topology considered here [41].

The scalar field and the electromagnetic potential are given by (8) and (9), respectively. Note that the scalar field and the electric potential cannot be switched off keeping the mass fixed. Thus, for fixed mass and electric charge, the theory described by the action (1) admits two branches of black hole solutions with the same geometry at the boundary: the hairy one presented here and the hyperbolic version of the Reissner-Nordström-AdS solution found in [5]. Both branches intersect only for $\mu = q = 0$, when the scalar field vanishes.

The causal structure of the electrically charges hairy black hole with hyperbolic horizon radically depends on the value of the mass as it is described below.

1. Hairy black hole with non-negative mass

Hairy black holes with positive mass, $\mu > 0$, possess a single event horizon located at

$$r_+ = \frac{l}{2} \left(1 + \sqrt{1 + 4G\mu/l} \right) , \tag{16}$$

satisfying $r_+ > l$, and the scalar field is regular everywhere.

For the massless black hole, $\mu = 0$, the scalar and electromagnetic field vanish, and the metric acquires a simple form

$$ds^2 = - \left(\frac{r^2}{l^2} - 1 \right) dt^2 + \left(\frac{r^2}{l^2} - 1 \right)^{-1} dr^2 + r^2 d\sigma^2 , \tag{17}$$

which corresponds to a negative constant curvature spacetime. It has been shown that this spacetime admits Killing spinors only if Σ belong to certain class of noncompact surfaces [42], and its stability has been analyzed in [43].

For non-negative mass, the causal structure coincides with the one for Schwarzschild-AdS black hole, but at each point of the Penrose diagram the sphere is replaced by the base manifold Σ.

2. Hairy black hole with negative mass: A black hole inside a black hole

It is well-known that the mass for black holes with hyperbolic horizons is bounded from below by a negative value [4, 5, 44]. The hairy black hole presented here shares the same feature, since in order to avoid the appearance of a naked singularity, the mass is bounded according to
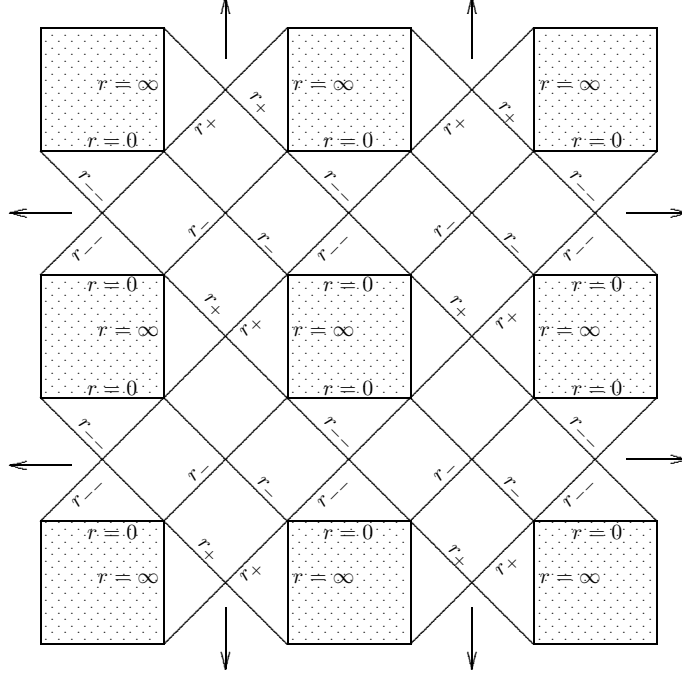
$$G\mu \geq -\frac{l}{4} . \tag{18}$$

FIG. 1: Penrose diagram for the generic case:$-l/4G < \mu < 0$. The dashed squared regions must be excised from the diagram.

For the generic case, i.e., when the mass is within the range $-l/4G < \mu < 0$, the hairy black hole possesses three horizons located at

$$r_{--} = \frac{l}{2}\left(-1 + \sqrt{1 - \frac{4G\mu}{l}}\right) ,$$

$$r_{-} = \frac{l}{2}\left(1 - \sqrt{1 + \frac{4G\mu}{l}}\right) ,$$

$$r_{+} = \frac{l}{2}\left(1 + \sqrt{1 + \frac{4G\mu}{l}}\right) ,$$

which satisfy $0 < r_{--} < -G\mu < r_{-} < \frac{l}{2} < r_{+}$. The outermost and innermost horizons, r_{+} and r_{--}, respectively are event horizons, and one can then say this metric describes "a black hole inside a black hole". The scalar field singularity, located at $-G\mu$, is timelike and lies between the region bounded by r_{--} and r_{-}, which is also surrounded by r_{+}. However, note that the geometry is regular at this timelike curve and hence, the stress-energy tensor is not singular there. The causal structure can be constructed following the general prescriptions indicated in [45] and [46]. The resulting Penrose diagram is shown in Fig. 1.

The causal structure shows that the outermost region is causally similar to the one for

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a Schwarzschild-AdS black hole. However, a timelike observer[3] who decide to cross the outermost even horizon could eventually have a less disastrous fate. Indeed at the first stage, the observer cannot avoid to go across r_-, but nevertheless, once he reaches the region between r_- and r_{--}, he is able to "switch on the rockets" and then eventually decide in what region he will last his last days. This region is bounded by the innermost event horizon r_{--}, and by r_-, which for him in practice is like a cosmological horizon. Note that the timelike singularity of the scalar field lies within this region, but as the geometry is regular there, test observers cannot feel it[4]. Therefore, the worst possibility for the pointlike observer is to fall into the inner black hole crossing r_{--}, inevitably finishing the trip in the future spacelike singularity at $r = 0$. Other possibility is to remain within the same region, i.e., to avoid falling into the inner black hole without going across r_-. In this case, the trip finishes at the timelike future infinity located at the corresponding bottom vertex of the excised squared region. The last possibility is to irreversibly escape from this region by going across r_-. In this case, the observer cannot avoid to trespass back r_+ reaching to a mirror copy of the original outermost region. The future infinity for a timelike observer in this region is located at the top of the corresponding upper vertex of the excised squared region. In sum, a timelike observer can go from one asymptotic region to a mirror copy of it without facing any kind of geometric singularity along the trip.

It is also worth to point that a radial null geodesic is able to travel from inside the inner white hole to $r = \infty$ in the outermost region.

The extremal case occurs for $G\mu = -l/4$, where the roots r_+ and r_- coalesce taking the value $r_+ = r_- = l/2 > -G\mu > r_{--} > 0$. Thus, in this case, the solution describes "a black hole inside an extremal black hole". Its Penrose diagram is shown in Fig. 2.

From the causal structure, one can see that for the extreme case, a timelike observer can also travel between two different asymptotic regions without seeing singularities in the geometry along the path.

Finally, for the case $G\mu < -l/4$, there is a single event horizon located at $r_{--} < -G\mu$, so that the geometric singularity at the origin is protected, but the timelike singularity of scalar field is naked since is accessible to observers in the asymptotic region. Note that,

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[3] We mean a test particle that follows a timelike curve.

[4] However, test fields that couple to the scalar field are expected to have an ill-defined behavior there. Nonetheless, the scalar field singularity is surrounded by ouertmost event horizon r_+.
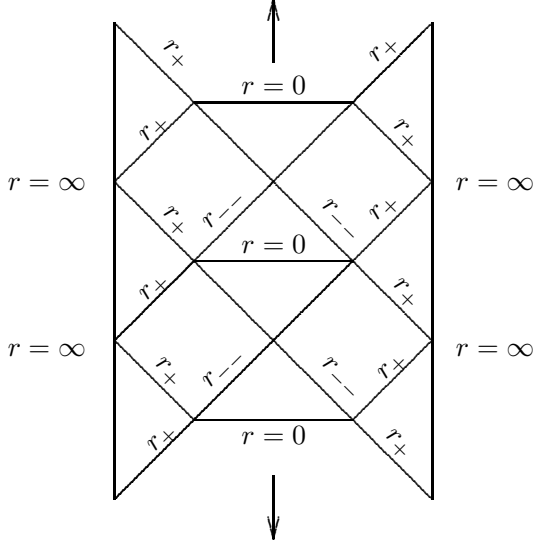
FIG. 2: Penrose diagram for $G\mu = -l/4$. This case corresponds to an extremal black hole.

although this singularity is not of geometrical origin, the expected ill behavior of fields that couple to the scalar field would be measured at the asymptotic region. In this sense, cosmic censorship is violated, and hence, this case should not be regarded as a physical solution.

3. Thermodynamics

For non negative masses, the non-extremal black hole solution discussed in the previous subsection possesses a single event horizon, and hence its thermodynamics can be suitably analyzed following Euclidean methods taking care of the nonminimal coupling of the scalar field with gravity. This approach can also be applied for the allowed range of negative masses under certain proper assumptions.

In the Euclidean method, the partition function for a thermodynamical ensemble is identified with the Euclidean path integral in the saddle point approximation around the Euclidean continuation of the classical solution [47]. The Euclidean continuation is obtained making $t \to it$, where the imaginary time has period β, which is identified with the inverse temperature. Thus, for a thermodynamical ensemble the action is related with free energy $\mathcal{F}$ as $I = \beta\mathcal{F}$.

In order to determine the suitable boundary terms that makes the Euclidean action to

be finite, it is enough to consider the minisuperspace of static Euclidean metrics

$$ds^2 = N(r)^2 f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 d\sigma^2, \tag{19}$$

with $0 \le t < \beta$, $r \ge r_+$. The scalar and electromagnetic fields are assumed to be of the form $\phi = \phi(r)$, $A = A_t(r)dt$, respectively. The reduced action principle is then

$$I = \frac{\beta \sigma}{4\pi} \int_{r_+}^{\infty} (N(r)\mathcal{H}(r) + A_t p') dr + B \ , \tag{20}$$

where B is a surface term, $p := \frac{r^2}{N} A_t'$, the symbol $'$ denotes d/dr, and the reduced Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = \frac{r^2}{2G}\left[8\pi G \left\{ \frac{1}{6} f^2 (\phi')^2 - \frac{1}{6}\phi\phi' \left[(f^2)' + \frac{4}{r} f^2 \right] - \frac{1}{3}\phi f^2 \phi'' + \alpha\phi^4 \right\} \right.$$
$$\left. + \left(1 - \frac{4\pi G}{3}\phi^2\right)\left(\frac{(f^2)'}{r} - \frac{1}{r^2}(-1 - f^2)\right) + \frac{Gp^2}{r^4} + \Lambda \right] . \tag{21}$$

Since the Euclidean solution satisfies the constraint $\mathcal{H} = 0$ and the Gauss law $p' = 0$, the action evaluated on the classical solution is just given by the boundary term B. In what follows, we work in the grand canonical ensemble, so that we consider variations of the action keeping fixed the temperature and the "voltage", i.e., β and $\Phi = A_t(\infty) - A_t(r_+)$ are constants without variation. The temperature is fixed requiring regularity of the metric at the horizon, which yields

$$\beta(N(r)(f^2(r))')\big|_{r=r_+} = 4\pi \ . \tag{22}$$

For the black hole (12) one obtains that

$$\beta^{-1} = \frac{1}{2\pi l}\left(\frac{2r_+}{l} - 1\right). \tag{23}$$

This boundary term is determined requiring the action (20) to attain extremum, i.e., $\delta I = 0$ within the class of fields considered here [48]. This implies that the variation of the boundary term is given by

$$\delta B = -\frac{\beta\sigma}{8\pi G}\left[Nr\left(1 - \frac{4\pi G}{3}\phi^2\right)\delta f^2 + Nr^2\frac{4\pi G}{3}\left[\phi\left((f^2)'\delta\phi - \phi'\delta f^2\right) + f^2\left(4\phi'\delta\phi - 2\phi\delta\phi'\right)\right]\right]_{r_+}^{\infty}$$
$$- \frac{\beta\sigma}{4\pi}A_t\delta p\Big|_{r_+}^{\infty} \equiv \delta B(\infty) - \delta B(r_+) . \tag{24}$$

By virtue of the field equations, one obtains that $p(r) = q$, and the "lapse" is chosen as $N(r) = 1$.

The variation of the fields at the horizon acquire the form

$$\delta p|_{r_+} = \delta q, \tag{25}$$

$$\delta f^2|_{r_+} = -(f^2)'|_{r_+} \delta r_+, \tag{26}$$

$$\delta \phi|_{r_+} = \delta \phi(r_+) - \phi'|_{r_+} \delta r_+. \tag{27}$$

It is useful to define the "effective Newton constant" $\tilde{G}$ as

$$\frac{1}{\tilde{G}} = \frac{1}{G}\left(1 - \frac{4\pi G}{3}\phi^2\right). \tag{28}$$

Thus, replacing (25)-(27) in (24), we obtain that the surface term at the horizon is

$$\delta B(r_+) = \delta\left(\frac{A_+}{4\tilde{G}_+}\right) - \beta\frac{\sigma}{4\pi}[A_t(r_+)]\delta q, \tag{29}$$

where $A_+ = \sigma r_+^2$ is the horizon area, and $\tilde{G}_+$ is the effective Newton constant at the horizon. The boundary term can then be integrated as

$$B(r_+) = \frac{A_+}{4\tilde{G}_+} + \frac{\beta\sigma}{4\pi}\Phi q. \tag{30}$$

The variation of fields at infinity are

$$\delta p|_\infty = \delta q, \tag{31}$$

$$\delta f^2|_\infty = -\frac{2G\delta\mu}{r} + O(r^{-2}), \tag{32}$$

$$\delta\phi|_\infty = \sqrt{-\frac{\Lambda}{6\alpha}}\left(\frac{1}{r} - \frac{2G\mu}{r^2}\right)G\delta\mu + O(r^{-3}), \tag{33}$$

$$\delta\phi'|_\infty = -\sqrt{-\frac{\Lambda}{6\alpha}}\left(\frac{1}{r^2} - \frac{4G\mu}{r^3}\right)G\delta\mu + O(r^{-4}), \tag{34}$$

from which one is able to determine the remaining boundary term. Using Eq. (24), the variation of the surface term at infinity is found to be

$$\delta B(\infty) = \frac{\beta\sigma}{4\pi}\delta\mu, \tag{35}$$

which, since that β is fixed, can be integrated as

$$B(\infty) = \frac{\beta\sigma}{4\pi}\mu. \tag{36}$$

Finally, the finite Euclidean action reads

$$I = B(\infty) - B(r_+) = \frac{\beta\sigma}{4\pi}\mu - \frac{A_+}{4\tilde{G}_+} - \frac{\beta\sigma}{4\pi}\Phi q, \tag{37}$$

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up to an arbitrary additive constant.

Since the Euclidean action is related with the free energy in the grand canonical ensemble, the mass M, the electric charge Q and the entropy S can be found using the familiar thermodynamical relations:

$$M = \left(\frac{\partial}{\partial\beta} - \beta^{-1}\Phi\frac{\partial}{\partial\Phi}\right)I = \frac{\sigma}{4\pi}\mu, \tag{38}$$

$$Q = -\beta^{-1}\frac{\partial I}{\partial\Phi} = \frac{\sigma}{4\pi}q, \tag{39}$$

$$S = \left(\beta\frac{\partial}{\partial\beta} - 1\right)I = \frac{A_+}{4\tilde{G}_+}. \tag{40}$$

Note that due to the nonminimal coupling of the scalar field with spacetime geometry, the entropy does not satisfy the area law, as in Refs. [20, 21, 49].

For the neutral black hole ($Q = 0$), since the effective Newton constant satisfy $\tilde{G}_+ > 0$, the entropy is always positive. However, for the electrically charged black hole, the positivity of the entropy implies that the mass must be bounded according to

$$-\frac{\sqrt{\frac{3l^2\alpha}{2\pi G}}}{(1 + \sqrt{\frac{3l^2\alpha}{2\pi G}})^2} < \frac{\mu G}{l} < \frac{\sqrt{\frac{3l^2\alpha}{2\pi G}}}{(1 - \sqrt{\frac{3l^2\alpha}{2\pi G}})^2}. \tag{41}$$

It is worth to point out that in general, if the mass is in the allowed range (18), but does not satisfy the bound (41), the effective Newton constant at the horizon becomes negative, which although the geometry is regular on and outside the event horizon, would lead to an unphysical negative entropy. This bound, has a natural geometric interpretation in the Einstein frame as explained in section V.

For the black holes with negative mass some comments are in order, since as the solution was found to describe "a black hole inside a black hole", we are facing a situation where the object possesses two event horizons, and therefore, extrapolation of the thermodynamical quantities found here must be taken with care. Indeed, dealing with the thermodynamics of black holes possessing more than one radiating horizon, as for the Schwarzschild-de Sitter solution, is a subtle issue, and some results have been obtained considering either of the two horizons as a boundary (for a recent treatment see Ref. [50]). For negative masses, as discussed previously, the causal structure shows that radial null geodesics are able to travel from inside the inner white hole to the outermost region, so that in principle, observers in the asymptotic region would be able to receive an additional contribution to the Hawking

radiation coming from the inner black hole if any. This issue is out of the scope of this work, and we only consider the thermodynamics associated to the Hawking radiation coming from the outermost event horizon. Under this assumption, the Euclidean approach works properly once the inner region is removed.

B.   Hairy gravitationally stealth solution with locally flat base manifold $(\gamma = 0)$

Let us consider now the case when the base manifold is of vanishing curvature, i.e., for $\gamma = 0$. In this case, the electromagnetic field vanishes, and when the cosmological constant is negative $(\Lambda = -3l^{-2})$, the metric (7) acquires the form

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + r^2 d\sigma^2 \ , \tag{42}$$

where the surface Σ is locally isometric to flat space $\mathbb{R}^2$. The scalar field is given by Eq. (13), which is real provided α is positive. In this case, the mass vanishes and hence the integration constant μ is a "pure hair" modulus parameter. The scalar field is regular everywhere for positive μ.

This solution is the four-dimensional generalization of the "nullnut" solution found in Ref. [51] in the conformal frame. Indeed, in this case the metric is locally AdS spacetime, and since the timelike Killing vector vanishes at the origin, it has nut on the null curve $r = 0$. Its causal structure is then the same as the one for the massless BTZ black hole [52].

As it occurs for its three-dimensional counterpart, the stress-energy tensor vanishes. The effect of having nontrivial matter fields with a vanishing total energy momentum tensor have also been discussed for flat spacetime in [53, 54] and for three-dimensional gravity with negative cosmological constant in Refs. [51, 55]. This effect has also been discussed for different setups in Refs. [56]. In the next Section, new solutions of this sort are found when the self-interaction coupling constant vanishes.

IV.   GRAVITATIONALLY STEALTH SOLUTIONS WITH A VANISHING SELF-INTERACTION COUPLING CONSTANT $(\alpha = 0)$

In the case of vanishing self-interaction coupling constant, i.e., for $\alpha = 0$, we find a massless solution dressed with a non-trivial scalar field. The metric reads

$$ds^2 = -\left[-\frac{\Lambda}{3}r^2 + \gamma\right]dt^2 + \left[-\frac{\Lambda}{3}r^2 + \gamma\right]^{-1}dr^2 + r^2 d\sigma^2 ,\tag{43}$$

the scalar field is given by

$$\phi = \frac{A}{r}\tag{44}$$

and the electromagnetic field is

$$F_{rt} = \partial_r A_t = \frac{q}{r^2} ,\tag{45}$$

where the electric charge is related to the integration constant A as

$$q^2 = -\frac{4\pi}{3}\gamma A^2 .$$

For this class of solutions, this last relation excludes the spherically symmetric case ($\gamma = 1$), and consequently, the metric is well behaved only for negative cosmological constant $\Lambda = -3l^{-2}$. Thus, the metric is a locally AdS spacetime, and therefore the solutions dress these spaces since they have a vanishing energy-momentum tensor with non-trivial matter fields.

When the curvature of the base manifold vanishes, i.e., for $\gamma = 0$, the metric coincides with the case discussed in the previous section (42), the electric charge vanishes, and the scalar field (44) is singular on the null curve $r = 0$.

In the case when the base manifold is of negative constant curvature $\gamma = -1$, the solution describes an electrically charged hairy black hole. The scalar and electric fields are given by (44), and (45), respectively, and the metric coincides with the one in Eq. (17), as in the case on nonvanishing self interaction coupling constant. However, in this case the scalar field (44) does not vanish, and its singularity at the origin is hidden by the event horizon located at $r_+ = l$.

The thermodynamics of this object can be readily found from the results obtained in section III A 3, and the mass, electric charge and the entropy are found to be,

$$M = 0 ,$$
$$Q = \frac{\sigma}{4\pi}q ,$$
$$S = \frac{A_+}{4\tilde{G}_+} = \frac{\sigma l^2}{4}\left(\frac{1}{G} - \frac{q^2}{l^2}\right) ,$$

respectively.

Note that if the electric and scalar fields were absent, the entropy for this black hole would be given by $S = \frac{\sigma l^2}{4G}$. This means that the Euclidean action, and hence the entropy, are sensitive to the presence of the gravitationally stealth hair.

The positivity of the entropy implies that the electric charge is bounded as

$$\frac{Gq^2}{l^2} < 1 \ . \tag{46}$$

If this bound is violated, the effective Newton constant at the horizon becomes negative. Note that although the geometry is regular on and outside the event horizon, this would yield an unphysical negative entropy. This bound, was obtained in [24] in the Einstein frame where it has a natural geometric interpretation.

V. MAPPING TO THE EINSTEIN FRAME: SOLUTIONS WITH A MINIMALLY COUPLED SELF-INTERACTING SCALAR FIELD

It is a well-known fact that there is a map between conformally and minimally coupled scalar fields. For the action principle considered here (1), the precise map is obtained as follows:

Performing a conformal transformation, followed of a scalar field redefinition of the form

$$\hat{g}_{\mu\nu} = \left(1 - \frac{4\pi G}{3}\phi^2\right) g_{\mu\nu} \ , \qquad \Psi = \sqrt{\frac{3}{4\pi G}} \tanh^{-1} \sqrt{\frac{4\pi G}{3}}\phi \ , \tag{47}$$

the action (1) is mapped to the Einstein frame

$$I[\hat{g}_{\mu\nu}, \Psi, A_\mu] = \int d^4x \sqrt{-\hat{g}} \left[\frac{\hat{R}}{16\pi G} - \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi - V(\Psi)\right] - \frac{1}{16\pi}\int d^4x\sqrt{-\hat{g}}F^{\mu\nu}F_{\mu\nu} \ , \tag{48}$$

where the self-interacting potential reads

$$V(\Psi) = \frac{\Lambda}{8\pi G}\left[\cosh^4\left(\sqrt{\frac{4\pi G}{3}}\Psi\right) + \frac{9\alpha}{2\pi G\Lambda}\sinh^4\left(\sqrt{\frac{4\pi G}{3}}\Psi\right)\right] \ .$$

Thus, any solution of the action in the conformal frame (1) can be mapped to a solution of the action in the Einstein frame (48) by means of the transformations (47). This map was discussed in Ref. [23], in the absence of the electromagnetic field.

Aplying (47) to the black hole solution given by Eqs. (12), (13), and (9) leads to an electrically charged hairy black hole for gravity with a minimally coupled self interacting

scalar field, which for the case of vanishing electric charge reduces to the one found in [23]. It can also be shown that the electrically charged black hole solution with a vanishing self interaction given by Eqs. (43) with $\Lambda = -3l^{-2}$, (44), and (45) can mapped to the one in Ref. [24] using (47) and a coordinate transformation.

Note that this map is invertible provided the conformal factor in Eq. (47) –which goes like the inverse of the effective Newton constant (28)– does not vanish. This means that the hypersurfaces in the conformal frame where the conformal factor vanishes are mapped into singularities of the geometry in the Einstein frame. Hence, in order to avoid dynamical instabilities, the region that one maps from the conformal to the Einstein frame is the one with positive effective Newton constant. Consequently, only a subset of well-behaved black hole solutions in the conformal frame can be mapped to well-behaved black holes in the Einstein frame. This subset is then defined by the black holes in the conformal frame for which the effective Newton constant becomes negative only inside the event horizon; otherwise the black hole would be mapped to a naked singularity in the Einstein frame. Therefore one concludes that the bounds for the black holes parameters in the conformal frame obtained from requiring the entropy to be positive, as in Eqs. (41) and (46), coincide with the ones that guarantee cosmic censorship in Einstein frame. This is due to the fact that the black hole entropy in the conformal frame is shown to be a quarter of the horizon area divided by the effective Newton constant at the horizon, which means that the class of causally well-behaved black hole solutions in the conformal frame which are regarded as unphysical because they would lead to a negative entropy, in the Einstein frame are mapped to spacetimes with naked singularities, which are discarded by cosmic censorship. It is worth pointing out that a dynamical argument in the conformal frame is mapped into a geometrical one in the Einstein frame.

As a final remark, it is worth pointing out that once the asymptotically locally AdS solutions presented here are mapped to the Einstein frame, they develop an asymptotic fall-off for the fields which is slower than the standard one, as in Ref. [57], for a localized distribution of matter. As discussed in [21], and further developed in [58, 59, 60], the presence of scalar fields with a slow fall off gives rise to a strong back reaction that relaxes the standard asymptotic form of the geometry, and it generates additional contributions to the charges that depend explicitly on the scalar fields at infinity which are not already present in the gravitational part. These effects has also been discussed recently following

different approaches in [61, 62, 63].

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